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THE SPACE OF CLOSED SUBGROUPS OF $\mathbf{R}^2$

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§1. INTRODUCTION

LET G be the set of closed additive subgroups of $\mathbf{R}^2$; each $\Gamma \in G$ is isomorphic to one of $\{0\}$, $\mathbf{Z}$, $\mathbf{R}$, $\mathbf{R}\times\mathbf{Z}$, $\mathbf{R}^2$ or $\mathbf{Z}^2$.

By identifying $\Gamma \in G$ with $\bar{\Gamma} = \Gamma \cup \{\infty\} \subset \mathbf{R}^2 \cup \{\infty\} = S^2$, we induce on G the Hausdorff metric on the space of closed subsets of S^2. For instance, in this topology, the subgroup generated by $(a, 0)$ converges to $\mathbf{R}\times\{0\}$ as a goes to 0 and to $\{0\}$ as a goes to infinity.

Let $H \subset G$ be the subset of those subgroups not isomorphic to $\mathbf{Z}^2$.

It is not difficult to show that G is a compact metric space which is in some sense four dimensional. The object of this paper is to identify G and H.

THEOREM. *The space G is homeomorphic to S^4 and H is homeomorphic to S^2.*

The pair (G, H) is homeomorphic to $(S(S^3), S(K))$, where S denotes suspension and $K \subset S^3$ is a trefoil knot.

In particular, we see that $H \subset G$ is knotted and not locally flat.

We know two proofs of the theorem, one based on Seifert fibrations and the other on elliptic functions. We shall give the latter here, in §3, 4 and 5.

The analogous space G_n of closed subgroups of $\mathbf{R}^n$ cannot be studied by the methods in this paper. Clearly, G_1 is a closed interval; G_n is not a manifold for $n \geq 3$. We do not know if G_n is always a suspension.

We wish to thank Adrien Douady for suggesting the problem (§2 is due to him) and Jean Lannes for pointing out that Proposition 1 was already known[3].

§2. THE SPACE H

A proof of the homeomorphism $H \cong S^2$ will come out of the proof of the main theorem, but it is easy to write out such a homeomorphism explicitly, as follows.

Identify $\mathbf{R}^2$ with $\mathbf{C}$ in the standard way. Parametrize S^2 by (φ, θ) with $0 \leq \varphi < 2\pi$ and $-\pi/2 \leq \theta \leq \pi/2$ and let P_1 and P_2 be the North and South poles, as in Fig. 1.

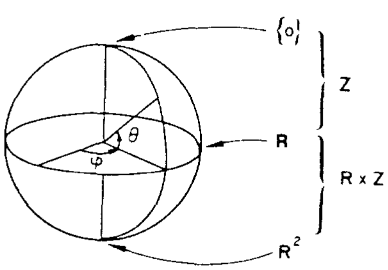

Fig. 1.

Map S^2 into H by the following prescription:

$$\begin{array}{rcl} P_1 & \text{maps to} & \{0\}, \\ (\varphi,\theta),\quad 0<\theta<\pi/2 & \text{maps to} & \mathbf{Z}\,(\tan\theta)\,e^{i\varphi/2}, \\ (\varphi,0) & \text{maps to} & \mathbf{R}\,e^{i\varphi/2}, \\ (\varphi,\theta),\quad -\pi/2<\theta<0 & \text{maps to} & \mathbf{R}\,e^{i\varphi/2}\times\mathbf{Z}\,(\mathrm{cotan}\,\theta)\,e^{i(\varphi+\pi)/2}, \\ P_2 & \text{maps to} & \mathbf{R}^2. \end{array}$$

We leave it to the reader to verify that this is indeed a homeomorphism.

§3. PERIODS OF ELLIPTIC CURVES

Let $\Sigma \subset \mathbf{C}^2$ be the surface of equation $a^3 - 27b^2 = 0$. For each $(a, b) \in \mathbf{C}^2 - \Sigma$, let $X_{(a,b)}$ be the plane curve of equation $y^2 = 4x^3 - ax - b$. Topologically, $X_{(a,b)}$ is a torus with a point removed, so $H_1(X_{(a,b)}) \cong \mathbf{Z}^2$. Let $\omega_{(a,b)} = dx/y$ restricted to $X_{(a,b)}$; it is a holomorphic one-form on $X_{(a,b)}$.

Let $\Gamma_{(a,b)}$ be the image of the map $H_1(X_{(a,b)}) \to \mathbf{C}$ given by $\gamma \mapsto \int_\gamma \omega_{(a,b)}$ and define $F: \mathbf{C}^2 - \Sigma \to G$ by $F(a, b) = \Gamma_{(a,b)}$.

PROPOSITION 1. *The map F: $\mathbf{C}^2 - \Sigma \to G$ is a homeomorphism onto the space of subgroups isomorphic to $\mathbf{Z}^2$.*

Proof. The inverse of F is given by the famous formula of Weierstrass[1, p. 267]. Let $\Gamma \in G$ be a lattice, then

$$F^{-1}(\Gamma) = \left(\frac{1}{60}\sum_{\substack{z\in\Gamma\\ z\neq 0}} \frac{1}{z^4}, \ \frac{1}{140}\sum_{\substack{z\in\Gamma\\ z\neq 0}} \frac{1}{z^6}\right). \quad \text{Q.E.D.}$$

LEMMA 2. *For any $t \in \mathbf{C} - \{0\}$, $\Gamma_{(t^2a, t^3b)} = t^{-1/2}\Gamma_{(a,b)}$.*

Proof. The map $x \mapsto tx$ lifts to an isomorphism f_t: $X_{(a,b)} \to X_{(t^2a,t^3b)}$, and $f_t^* \omega_{(t^2a,t^3b)} = t^{-1/2}\omega_{(a,b)}$. Q.E.D.

The next proposition is well known, under the name "degeneration of the period matrix" ([2], for instance), but it is easier to reprove it than to dig the exact statement we need out of the literature.

Let $(3t^2, -t^3)$ be a point in Σ.

PROPOSITION 3. *As (a, b) converges to $(3t^2, -t^3)$, $\Gamma_{(a,b)}$ converges to the group generated by $\pi i\sqrt{(2/3t)}$, or to $\{0\}$ if $t = 0$.*

Proof. In view of Lemma 2, we may assume $t = 1$. Pick $\epsilon < 1/2$; if (a, b) is sufficiently close to $(3, -1)$ then

$$4x^3 - ax - b = (2x - 2x_1)(2x - 2x_2)(x + x_1 + x_2),$$

with $|x_1 - 1/2| < \epsilon/2$ and $|x_2 - 1/2| < \epsilon/2$; let $\operatorname{Re} x_1 \leq \operatorname{Re} x_2$.

Let γ_1 and γ_2 be the paths in $\mathbf{C}$ drawn in Fig. 2. Let $\tilde{\gamma}_1$ be one of the two lifts of γ_1 to $X_{(a,b)}$ and $\tilde{\gamma}_2$ be a closed curve on $X_{(a,b)}$ which covers γ_2 on one sheet of $X_{(a,b)}$ and $-\gamma_2$ on the other. Then the intersection number $\tilde{\gamma}_1 \cdot \tilde{\gamma}_2 = \pm 1$, so $(\tilde{\gamma}_1, \tilde{\gamma}_2)$ forms a basis for $H_1(X_{(a,b)})$.

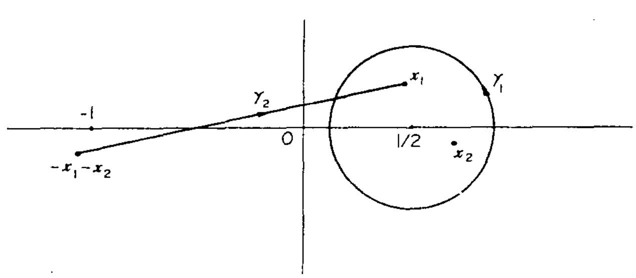

Fig. 2.

As $\epsilon \to 0$, $\omega_{(a,b)} \to dx/(2(x - 1/2)\sqrt{(x + 1)})$ which has a simple pole at 1/2; therefore

$$\int_{\tilde{\gamma}_1} \omega_{(a,b)} \to \pi i \sqrt{\frac{2}{3}},$$

by the calculus of residues.

Let

$$I = \int_{\tilde{\gamma}_2} \omega_{(a,b)} = 2\int_{\gamma_2} \frac{dx}{\sqrt{[(2x - 2x_1)(2x - 2x_2)(x + x_1 + x_2)]}}$$

be the other period; Lemma 3 will be proved if we show that $\operatorname{Re} I \to \infty$ as $\epsilon \to 0$.

Parametrize γ_2 by $\gamma(t) = (t-1)(x_1+x_2)+tx_1$; then

$$I = \int_0^1 \frac{\gamma'(t)\,\mathrm{d}t}{\sqrt{[(\gamma(t)-x_1)(\gamma(t)-x_2)(\gamma(t)+x_1+x_2)]}}$$

and the arguments of the terms under the integral sign are such that for one of the two continuous choices of the square root, the integrand always lies in the sector of angle $\pi/4$ around the positive real axis. Therefore

$$\operatorname{Re} I \geq \frac{1}{\sqrt{2}}\int_0^1 \left|\frac{\gamma'(t)}{\sqrt{[(\gamma(t)-x_1)(\gamma(t)-x_2)(\gamma(t)+x_1+x_2)]}}\right| \mathrm{d}t.$$

To estimate this integral, use $|\gamma(t)+x_1+x_2| \leq 2$ and $|\gamma(t)-x_2| \leq |\gamma(t)-x_1|+\epsilon$ and set $s = |\gamma(t)-x_1|$. This gives

$$\operatorname{Re} I \geq \frac{1}{2}\int_0^{|2x_1+x_2|} \frac{\mathrm{d}s}{\sqrt{[s(s+\epsilon)]}} = \frac{1}{2}\log\frac{1}{\epsilon}+0(1). \qquad \text{Q.E.D.}$$

In view of Proposition 3, set $\Gamma_{(3t^2,-t^3)} = \mathbf{Z}(\pi i\sqrt{(2/3t)})$ if $t \neq 0$ and $\Gamma_{(0,0)} = \{0\}$. The map $(a,b) \mapsto \Gamma_{(a,b)}$ is a homeomorphism of $\mathbf{C}^2$ onto the part of G consisting of discrete subgroups.

§4. THE SUBGROUPS ISOMORPHIC TO R

In this paragraph we will attach a circle to $\mathbf{C}^2$ "at infinity along Σ", corresponding to the subgroups isomorphic to $\mathbf{R}$.

Define the area function A: $\mathbf{C}^2 \to (0,\infty]$ by $A(a,b) = \text{Area}(\mathbf{C}/\Gamma_{(a,b)})$; in particular, $A(a,b) = \infty$ if $(a,b) \in \Sigma$.

For any $A \in (0,\infty]$, define φ_A: $[0,1] \to [0,\infty]$ by $\varphi_A(t) = t/(1-t+A^{-1})$; φ_A is a homeomorphism $[0,1] \to [0,A]$.

Let B be the unit ball in $\mathbf{C}^2$ for the Euclidean norm $|z|$, let S^3 be its boundary and let $K = S^3 \cap \Sigma$; clearly K is a trefoil knot.

Map $B-K$ into $\mathbf{C}^2$ by the formula

$$F\colon z = (z_1,z_2) \mapsto \left((\varphi_{A(z/|z|)}(|z|))^2\frac{z_1}{|z|}, (\varphi_{A(z/|z|)}(|z|))^3\frac{z_2}{|z|}\right).$$

PROPOSITION 4. (a) *The map F is a homeomorphism of $B-K$ onto*

$$\{(a,b) \in \mathbf{C}^2 | A(a,b) \geq 1;$$

(b) *If $(3t^2,-t^3) \in K$ (i.e. $9|t|^4+|t|^6 = 1$), then*

it takes the interior of the cone over K onto Σ.

$$\lim_{z\to(3t^2,-t^3)} \Gamma_{F(z)} = \mathbf{R}\left(i\pi\sqrt{\frac{2}{3t}}\right).$$

Proof. Part (a) is clear from the formula for F. For (b), let z_n be a sequence in $B-K$ converging to $(3t^2,-t^3)$ and let α_n be the unique positive numbers such that $F(\alpha_n z_n) = 1$; clearly $\alpha_n \to 0$ as $n \to \infty$.

By Lemma 2 and Proposition 3, $F(z_n)$ is a group one generator of which is near $i\pi\sqrt{(2\alpha_n/3t)}$; since $A(F(z_n)) \geq 1$, any other generator of $\Gamma_{F(z_n)}$ must be near infinity for large n. Q.E.D.

Remark. The fact that the sequence $F(z_n)$ stayed in the region $A(a,b) \geq 1$ is crucial to the proof.

COMPENDIUM. *Collecting Propositions* 1, 3 *and* 4, *we find that the map* $z \mapsto \Gamma_{F(z)}$ *extends to a homeomorphism of B onto the part of G formed of the groups isomorphic to* $\{0\}$, $\mathbf{Z}$ *and* $\mathbf{R}$, *and those lattices* Γ *such that* $\text{Area}(\mathbf{C}/\Gamma) \geq 1$.

Moreover $K \subset S^3$ corresponds to the subgroups isomorphic to $\mathbf{R}$, *the interior of the cone over K corresponds to the groups isomorphic to* $\mathbf{Z}$, *with* 0 *corresponding to the group* $\{0\}$, *and S^3-K corresponds to the lattices* Γ *such that* $\text{Area}(\mathbf{C}/\Gamma) = 1$.

§5. THE SUSPENSION INVOLUTION

In this paragraph we will exhibit an involution of G corresponding to the canonical involution of a suspension.

For any $\Gamma \in G$, let $\Gamma' = \{z \in \mathbf{C} \mid \mathrm{Im}\, \bar{z}w \in \mathbf{Z}$ for all $w \in \Gamma\}$. A simple computation shows that:

$$
\begin{array}{lll}
\text{if} & \Gamma = \{0\}, & \Gamma' = \mathbf{R}^2; \\
\text{if} & \Gamma = \mathbf{Z}w, & \Gamma' = \mathbf{R}w + \mathbf{Z}(i/\bar{w}); \\
\text{if} & \Gamma = \mathbf{R}w, & \Gamma' = \Gamma = \mathbf{R}w; \\
\text{if} & \Gamma = \mathbf{Z}w_1 + \mathbf{Z}w_2, & \Gamma' = \mathbf{Z}\dfrac{w_1}{\mathrm{Im}\,(\bar{w}_1 w_2)} + \mathbf{Z}\dfrac{w_2}{\mathrm{Im}\,(\bar{w}_1 w_2)}.
\end{array}
$$

In particular, this involution corresponds to $(\varphi, \theta) \mapsto (\varphi, -\theta)$ in the parametrization of H described in §2.

Proof of the theorem. Since $\mathrm{Im}\,(\bar{w}_1 w_2) = \mathrm{area}\,(\mathbf{Z}w_1 + \mathbf{Z}w_2)$, we see that the fixed locus of the involution is formed of the groups isomorphic to $\mathbf{R}$ and the lattices Γ such that Area $(\mathbf{C}/\Gamma) = 1$.

Putting this together with the compendium at the end of §4, we see that G is homeomorphic to the union of two copies of B, glued by the identity along their common boundary S^3.

Moreover, the trefoil knot $K \subset S^3$ corresponds to the groups isomorphic to $\mathbf{R}$ and the cone over K in one of the balls corresponds to the groups isomorphic to $\mathbf{Z}$, with 0 at the vertex, whereas the cone over K in the other ball corresponds to the groups isomorphic to $\mathbf{R} \times \mathbf{Z}$, with $\mathbf{R}^2$ at the vertex. This proves the theorem.

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